The Quantum as an Emergent System

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*a system which maps order of its environment onto its own organization (behaviour)*

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“It is not enough to know the ‘fundamental’ laws at a given level. It is the solutions to equations, not the equations themselves, that provide a mathematical description of the physical phenomena. ‘Emergence’ refers to properties of the solutions – in particular, the properties that are not readily apparent from the equations.”

Sam Schweber, in Physics Today (1993)
The "enlightenment task": “...trying to explain the unnatural by the natural - in this case, the ‘unnatural’ being quantum physics and the ‘natural’ being classical physics...”

Philip Pearle (2005), in: Book Review of S. L. Adler, *Quantum theory as an emergent phenomenon*
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“Classical particles and their dynamics are re-introduced, but a strong element of the unnatural remains. In the deBroglie-Bohm and Madelung models, it is the mysterious quantum force. In the Nelson model, it is the mysterious backward diffusion process (which, together with the usual classical forward diffusion process, forces a particle’s drift – its mean position – to be a dynamically determined quantity instead of, as classically, an independent variable set by external influences).

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Proposal:

A quantum system = an emergent system
= a self-organizing, dynamical entity whose complete description needs more than just one “basic” level
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Explanatory framework:

“21st century classical physics”
(ActionEvent Equilibrium Thermodynamics, Ballistic Diffusion, Diffusion Wave Fields, ...)
and a specific model, in analogy to...
bouncers and walkers


Nonequilibrium Thermodynamics:

Quantum "particle" of energy $E = \hbar \omega$

understood as off-equilibrium steady state oscillation maintained by a constant throughput of energy provided by the ("classical") zero-point energy field
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"natural drift": absorption and re-emission of kinetic energy (→ "ballistic diffusion")
Nonequilibrium Thermodynamics:

Quantum "particle" of energy $E = \hbar \omega$

understood as off-equilibrium steady state oscillation maintained by a constant throughput of energy provided by the ("classical") zero-point energy field,

i.e., a quantum emerges from the synchronized dynamical coupling between a "bouncer" and its wave-like environment.
AINS-Derivations from “21st century classical physics” of the following quantum mechanical features:

- Planck’s relation for the energy of a particle, \( E = \hbar \omega \)
  
  *Found. Phys. 41, 9 (2011) 1437-1453*

- the exact (one- and \( n \)-particle) Schrödinger equation for conservative and non-conservative systems,
  

- the Heisenberg uncertainty relations,
  

- the quantum mechanical superposition principle and Born’s rule,
  
  *Physica A 389 (2010) 4473-4484*

- the quantum mechanical decay of a Gaussian wave packet (ibid.)

- quantum mechanical interference at the double slit
  

- Further, it was proven that free quantum motion exactly equals sub-quantum anomalous (i.e., “ballistic”) diffusion.
  
Contents of this talk:

- Sketching derivation of exact Schrödinger equation (i.e., from 3 “classical” assumptions)

- “thermodynamic” re-formulation of de Broglie-Bohm theory

- Gaussian wave packet dispersion: free quantum motion exactly equals sub-quantum anomalous (i.e., “ballistic”) diffusion

- “the path excitation field”: interference at the double slit

- outlook
proposition of emergence:

behaviour of phase space distribution functions $f$ on the sub-quantum level (with temperature $T_0$) mirrored by configuration space probability densities $P$ on the quantum level:

$$\frac{f(\Gamma(t), 0)}{f(\Gamma(0), 0)} = e^{-\frac{\Delta Q}{kT_0}} \equiv \frac{P(x, t)}{P(x, 0)}$$

exact result from nonequilibrium thermodynamics for a "steady state" system (Williams, Evans, et al.);
equilibrium-type behaviour (!)
proposition of emergence:

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Relevant description of the system no longer given by the totality of all coordinates and momenta of microscopic entities, but reduced to only the particle coordinates.
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\]

Relevant description of the system no longer given by the totality of all coordinates and momenta of microscopic entities, but reduced to only the particle coordinates.

The many microscopic degrees of freedom (subquantum domain): *recast* into more “macroscopic” properties (collective wave-like behaviour) on the quantum level, i.e., with

\[
P(x,t) = R^2(x,t)
\]
Definition of the total energy of a "particle" and its thermal bath:

\[ E_{\text{tot}} = \hbar \omega + \frac{kT_0}{2}, \quad \text{where} \quad \frac{kT_0}{2} = \frac{\hbar \omega}{2} = \frac{mu^2}{2} = \frac{(\delta p)^2}{2m} \quad \text{(p.d.o.f.)} \]
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Boltzmann’s formula

\[ \Delta Q = 2\omega \delta S = 2\omega \left[ \delta S (t) - \delta S (0) \right] \]
provides

\[ P(x,t) = P(x,0) e^{\frac{-\Delta Q}{\hbar \omega}} = P(x,0) e^{\frac{2}{\hbar} \delta S} \]
and thus

\[ \delta p(x,t) = mu(x,t) = \nabla \left( \delta S (x,t) \right) = -\frac{\hbar}{2} \frac{\nabla P(x,t)}{P(x,t)} = -\hbar \frac{\nabla R(x,t)}{R(x,t)} \]
Definition of the total energy of a “particle“ and its thermal bath:

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provides

\[ P(x,t) = P(x,0) e^{\frac{-\Delta Q}{\hbar \omega}} = P(x,0) e^{\frac{-2\delta S}{\hbar}} \]

and thus

\[ \delta p(x,t) = mu(x,t) = \nabla \left( \delta S(x,t) \right) = -\frac{\hbar \nabla P(x,t)}{2 P(x,t)} = -\hbar \frac{\nabla R(x,t)}{R(x,t)} \]

into action integral

\[ \int L d^n x dt = \int P(x,t) \left[ \frac{\partial S}{\partial t} + V + \frac{1}{2m} \nabla S \cdot \nabla S + \frac{1}{2m} \nabla \left( \delta S \right) \cdot \nabla \left( \delta S \right) \right] d^n x dt \]
Average Orthogonality Condition (AOC):

\[ \mathbf{p} \cdot \delta \mathbf{p} := \int P(x, t) \mathbf{p} \cdot \delta \mathbf{p} \ d^n x = 0. \]

Thus,

\[ A = \int P \left[ \frac{\partial S}{\partial t} + V + \frac{P_{\text{tot}}^2}{2m} \right] d^n x dt, \]

with

\[ P_{\text{tot}} = \mathbf{p} + \delta \mathbf{p} = \nabla (S + \delta S) =: \hbar \mathbf{k} + \hbar \mathbf{k}_u = \nabla S - \hbar \frac{\nabla R}{R}. \]
Average Orthogonality Condition (AOC):

\[ \overline{p \cdot \delta p} := \int P(x,t) \, p \cdot \delta p \, d^n x = 0. \]

Thus,

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With "Madelung transformation" \( \psi^{(*)} = \text{Re} \left( \frac{-iS}{\hbar} \right), \ P = R^2 = |\psi|^2, \)

\[ p_{tot}^2 = p^2 + (\delta p)^2 = \hbar^2 \left[ \left( \frac{\nabla S}{\hbar} \right)^2 + \left( \frac{\nabla R}{R} \right)^2 \right] = \hbar^2 \left| \frac{\nabla \psi}{\psi} \right|^2 \]

= the "vocabulary", i.e., for translations between sub-quantum and quantum "languages"
**Average Orthogonality Condition (AOC):**

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\[ \Rightarrow \int L dt = \int d^n x dt \left[ |\psi|^2 \left( \frac{\partial S}{\partial t} + V \right) + \frac{\hbar^2}{2m} |\nabla \psi|^2 \right] \Rightarrow \text{Schrödinger equation} \]

(Note: no de Broglie-Bohm-type Hamilton-Jacobi Eq. necessary!)
Illustration of Average Orthogonality Condition:

\[ k_u = \frac{\nabla(\delta S)}{\hbar} = -\frac{\nabla R}{R} \]

\[ S = \text{constant} \]

**classical Hamiltonian flow:**

\[ \nabla \cdot \mathbf{v} = 0 \]

\[ \mathbf{k} \cdot \mathbf{k}_u = 0 \]

**quantum flow:**

\[ \int P (\nabla \cdot \mathbf{v}) \, d^n x = 0 \]

\[ \int P (\mathbf{k}_u \cdot \mathbf{k}) \, d^n x = 0 \]
“Bohmian version“:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + U = 0,$$
with "quantum potential" $U$,

$$U = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla P}{P} \right)^2 - \frac{\nabla^2 P}{P} \right] = \frac{mu \cdot u}{2} - \frac{\hbar}{2} (\nabla \cdot u).$$

Then,

$$u = -\frac{\hbar}{2m} \frac{\nabla P}{P} = \frac{2}{2\omega m} \nabla Q$$

provides the thermodynamic formulation of the “quantum potential”:

$$U = \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla Q}{\hbar \omega} \right)^2 - \frac{\nabla^2 Q}{\hbar \omega} \right].$$
a vanishing “quantum potential”, $U=0$, is identical to the classical diffusion (heat) equation:

$$U = 0 \iff \nabla^2 Q - \frac{1}{D} \frac{\partial Q}{\partial t} = 0.$$  

I.e., even for free (single path) particles, one can identify a “heat dissipation” process emanating from the particle.
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⇒ a non-vanishing “quantum potential” is a means to describe the spatial and temporal dependencies of the corresponding thermal flow in the case that the particle is not free (e.g., many paths via Gaussian).

$$U \neq 0 \iff \nabla^2 Q - \frac{1}{D} \frac{\partial Q}{\partial t} = q(x) e^{i\omega t}.$$
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$$U \neq 0 \iff \nabla^2 Q - \frac{1}{D} \frac{\partial Q}{\partial t} = q(x)e^{i\omega t}. \text{ …when Fourier-trf., becomes}$$

$$\nabla^2 \tilde{Q}(x, \omega) - \kappa^2 \tilde{Q}(x, \omega) = Q(x, \omega),$$

i.e., the defining equation for (nonlocal!) Diffusion Wave Fields (Mandelis et al.: “entire domain ‘breathes’ in phase with oscillator” )
Sub-Quantum Mechanics:

simple solutions for
* dispersion of a Gaussian wavepacket and
* double slit interference

- no use of quantum mechanics, complex numbers, etc.
- simple calculations, exact agreement with qu. m. results
- at the end: “translation” with aid of the “vocabulary”
  → familiar qu. m. equations and solutions
Decay of a Gaussian wavepacket: anomalous diffusion

From $E_{\text{tot}} = \hbar \omega + \left(\frac{\delta p}{2m}\right)^2$ = const., with AOC and initial Gaussian distribution,

$u_0 = -\frac{\hbar}{2m \sigma} \frac{\nabla P}{P} = \frac{D}{\sigma_0}$ : $\overline{x^2} = x^2|_{t=0} + u_0 t^2$

and thus also $\sigma^2 = \sigma_0^2 \left(1 + \frac{D^2 t^2}{\sigma_0^4}\right)$. 
Decay of a Gaussian wavepacket: anomalous diffusion

From \( \overline{E_{\text{tot}}} = \hbar \omega + \frac{mu^2}{2} = \text{const.} \), with AOC and initial Gaussian distribution, \( u_0 = -\frac{\hbar}{2m} \frac{\nabla \overline{P}}{\overline{P}} = \frac{D}{\sigma_0} \) : \( x^2 = x^2|_{t=0} + u_0^2 t^2 \) and thus also \( \sigma^2 = \sigma_0^2 \left( 1 + \frac{D^2 t^2}{\sigma_0^4} \right) \).

... results from momentum fluctuations (up to second order) \( \pm m (u \pm \delta u) \); hence, the "natural" drift \( x(t) = x(0) \pm (u \pm \delta u) t \).

with AOC: \( \overline{x^2}(t) = \overline{x^2}(0) + \left[ \overline{u^2} + (\delta u)^2 \right] t^2 \) \( u_0^2 \)}
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\[
\sigma_0^2 = \sigma_0^2 \left( 1 + \frac{D^2 t^2}{\sigma_0^2} \right).
\]

Define \( D(t) = u_0^2 t \), and for \( t \gg \omega^{-1} \), \( \langle D(t) \rangle := \frac{1}{t} \int_0^t D(t') dt' = D(t)/2 \), then

\[
\bar{x}^2 = x^2|_{t=0} + 2 \langle D(t) \rangle t
\]

... a Brownian-type displacement with a time-dependent diffusivity ("ballistic diffusion")

time reversible, dependent on initial conditions (\( u_0 \)), etc. (Klafter, Vainstein, et al.)
Decay of a Gaussian wavepacket: anomalous diffusion

From \( \overline{E}_{\text{tot}} = \hbar \omega + \frac{(\delta p)^2}{2m} \) = const., with AOC and initial Gaussian distribution, \( u_0 = -\frac{\hbar}{2m} \frac{\nabla P}{P} = \frac{D}{\sigma_0} \) : \( \overline{x^2} = \overline{x^2}_{t=0} + u_0^2 t^2 \) and thus also \( \sigma^2 = \sigma_0^2 \left( 1 + \frac{D^2 t^2}{\sigma_0^4} \right) \).

Define \( D(t) = u_0^2 t \), and for \( t \gg 1/\omega \), \( \langle D(t) \rangle := \frac{1}{t} \int_0^t D(t') dt' = D(t)/2 \), then

\[ \overline{x^2} = \overline{x^2}_{t=0} + 2 \langle D(t) \rangle t \]

... a Brownian-type displacement with a time-dependent diffusivity ("ballistic diffusion").

→ average total velocity field of a Gaussian wave packet:

\[ v_{\text{tot}}(t) = v(t) + \left[ \overline{x_{\text{tot}}(t)} - v t \right] \frac{D(t)}{\sigma^2} \]
Computer simulations with Coupled Map Lattices (CML):

\[ P[i, k + 1] = P[i, k] + \frac{D[k + 1]}{\Delta x^2} \Delta t \left\{ P[i + 1, k] - 2P[i, k] + P[i - 1, k] \right\}, \text{ with } D \rightarrow D(t) \]

\[ D(t) = u_0^2 t = \frac{D^2}{\sigma_0^2} t \]
Computer simulations with Coupled Map Lattices (CML):

\[
P[i, k+1] = P[i, k] + \frac{D[k+1] \Delta t}{\Delta x^2} \{P[i+1, k] - 2P[i, k] + P[i-1, k]\}, \text{ with } D \rightarrow D(t)
\]

\[
D(t) = u_0^2 t = \frac{D^2}{\sigma_0^2} t
\]

meaning of *time-dependent diffusivity*:

* reflects a changing thermal environment

* dissipation of “heat” from a constrained (i.e., by the slit potential) to an unconstrained vacuum
Dispersion of a free Gaussian wave-packet in a classical (CML) simulation
Dispersion of a free Gaussian wave-packet in a classical (CML) simulation
particle acceleration in a linear potential:

\[ \ddot{x}_{\text{tot}}(t) = -g + \frac{x(0)\hbar^2}{4m^2\sigma_0\sigma^3} = -g + \frac{x(0)\hbar^2}{4m^2\sigma_0^4} \left[ 1 + \frac{\hbar^2t^2}{4m^2\sigma_0^4} \right]^{-3/2} \]

\[ = -g + x(0)\frac{u_0^2}{\sigma_0^2} \left[ 1 + \frac{u_0^2t^2}{\sigma_0^2} \right]^{-3/2} \]
The “Path Excitation Field”

Couder et al.: “memory driven trajectories” (i.e., past $\rightarrow$ present)

$\Rightarrow$ imply also a “path excitation field” (i.e., present $\rightarrow$ future)
composed of vector fields $\mathbf{v}$ and $\mathbf{u}$
The “Path Excitation Field”

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→ imply also a “path excitation field” (i.e., present → future)
composed of vector fields \( \mathbf{v} \) and \( \mathbf{u} \)

With \( \mathbf{u}_\alpha \) representing ensemble of hypothetical sub-quantum motions
in small volume around \( \mathbf{x} \), its mean

\[
\mathbf{u}(\mathbf{x},t) = \frac{1}{N(\mathbf{x},t)} \sum_{\alpha=1}^{N(\mathbf{x},t)} \mathbf{u}_\alpha(t) = \frac{1}{2N} \left[ \sum_{\alpha=1}^{N} \mathbf{u}_{\alpha,+}(t) + \sum_{\alpha=1}^{N} \mathbf{u}_{\alpha,-}(t) \right] = \frac{1}{2} \left[ \mathbf{u}_+ + \mathbf{u}_- \right]
\]

refers to Brownian motion of very large number \( N \) of a bouncer’s
c\textit{possible path directions} - due to existence of wave-like “excitations”
of zero-point field;
The “Path Excitation Field”

With $u_{\alpha}$ representing ensemble of hypothetical sub-quantum motions in small volume around $x$, its mean

$$u(x,t) = \frac{1}{N(x,t)} \sum_{\alpha=1}^{N(x,t)} u_{\alpha}(t) = \frac{1}{2N} \left[ \sum_{\alpha=1}^{N} u_{\alpha,+}(t) + \sum_{\alpha=1}^{N} u_{\alpha,-}(t) \right] = \frac{1}{2} [u_+ + u_-]$$

refers to Brownian motion of very large number $N$ of a bouncer’s possible path directions - due to existence of wave-like “excitations” of zero-point field; averaging provides a “smoothed-out”

average velocity field  \[ \bar{u}(x,t) = \int P u(x,t) d^n x \]

also: \[ \bar{v}(x,t) = \int P v(x,t) d^n x \]

and \[ \bar{v} \cdot \bar{u} = \int P \bar{v} \cdot \bar{u} d^n x = 0 \] (average orthogonality!)
The “Path Excitation Field”

average velocity fields \( \overline{u(x,t)} = \int P\overline{u(x,t)} \, d^n x, \overline{v(x,t)} = \int P\overline{v(x,t)} \, d^n x \)

with \( \overline{v \cdot u} = \int P \overline{v \cdot u} \, d^n x = 0 \)

confluence of \( i \) possible paths: \( \overline{v_{\text{tot}}} = \sum_i \overline{v_{\text{tot},i}}, \) where

\[ \overline{v_{\text{tot},i}} = \overline{v_i} + \frac{\overline{u_{i+}}}{2} + \frac{\overline{u_{i-}}}{2} \]
The “Path Excitation Field”

average velocity fields \( \overline{u(x,t)} = \int P u(x,t) d^n x, \quad \overline{v(x,t)} = \int P v(x,t) d^n x \)

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\]

→ example of double slit interference:

very simple calculation of intensity distributions, particle trajectories, and probability density currents

* without any reference to quantum theory
* using only high school math!
\[
\frac{\pi}{2} + \varphi = \arccos (\hat{v}_1 \cdot \hat{u}_2)
\]

\[
\cos \left( \frac{\pi}{2} \pm \varphi \right) = \mp \sin \varphi
\]
Superposition of classical plane waves: \( R_{\text{tot}} \mathbf{k} = R_1 \mathbf{k}_1 + R_2 \mathbf{k}_2 \); with \( |\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}| \):

\[
P_{\text{tot}} := R_{\text{tot}}^2 = \left| R_1 \hat{\mathbf{k}}_1 + R_2 \hat{\mathbf{k}}_2 \right|^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos \varphi = P_1 + P_2 + 2\sqrt{P_1P_2} \cos \varphi
\]

phase difference: a) relative motion of w.p. and b) dispersion term:

\[
\varphi = \frac{1}{\hbar} (m \Delta v_{\text{tot, } x} x) = 2m v_x \frac{x}{\hbar} - (X + v_x t) x \frac{D(t)}{D\sigma^2}
\]
Superposition of classical plane waves: $R_{\text{tot}} k = R_1 k_1 + R_2 k_2$; with $|k_1| = |k_2| = |k|$

$$P_{\text{tot}} := R_{\text{tot}}^2 = \left| R_1 \hat{k}_1 + R_2 \hat{k}_2 \right|^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos \varphi = P_1 + P_2 + 2\sqrt{P_1P_2} \cos \varphi$$

phase difference: a) relative motion of w.p. and b) dispersion term:

$$\varphi = \frac{1}{\hbar} \left( m \Delta v_{\text{tot}, x} x \right) = 2m v_x \frac{x}{\hbar} - (X + v_x t) x \frac{D(t)}{D\sigma^2}$$

Geometric meaning of the “path excitation field”:

$$R_{\text{tot}} \overrightarrow{v_{\text{tot}}} = R_1 \overrightarrow{v_{\text{tot},1}} + R_2 \overrightarrow{v_{\text{tot},2}} \quad \text{with} \quad \overrightarrow{v_{\text{tot},i}} = \overrightarrow{v_i} + \frac{\overrightarrow{u_{i+}}}{2} + \frac{\overrightarrow{u_{i-}}}{2} \quad \text{into} \quad \overrightarrow{J_{\text{tot}}} = R_{\text{tot}}^2 \overrightarrow{v_{\text{tot}}}$$

$$\rightarrow \overrightarrow{J_{\text{tot}}} = R_1^2 \overrightarrow{v_1} + R_2^2 \overrightarrow{v_2} + R_1R_2 \begin{pmatrix} \left( \overrightarrow{v}_1 + \overrightarrow{v}_2 \right) \cos \left( \overrightarrow{v}_1, \overrightarrow{v}_2 \right) + \left( \overrightarrow{v}_1 + \frac{\overrightarrow{u}_2}{2} \right) \cos \left( \overrightarrow{v}_1, \overrightarrow{u}_2 \right) - \left( \overrightarrow{v}_1 - \frac{\overrightarrow{u}_2}{2} \right) \cos \left( \overrightarrow{v}_1, \overrightarrow{u}_2 \right) \\ + \frac{\overrightarrow{u}_1}{2} + \overrightarrow{v}_2 \right) \cos \left( \overrightarrow{u}_1, \overrightarrow{v}_2 \right) - \left( \frac{\overrightarrow{u}_1}{2} + \overrightarrow{v}_2 \right) \cos \left( \overrightarrow{u}_1, \overrightarrow{v}_2 \right) + \left( \frac{\overrightarrow{u}_1}{2} + \frac{\overrightarrow{u}_2}{2} \right) \cos \left( \overrightarrow{u}_1, \overrightarrow{u}_2 \right) \\ - \left( \frac{\overrightarrow{u}_1}{2} - \frac{\overrightarrow{u}_2}{2} \right) \cos \left( \overrightarrow{u}_1, \overrightarrow{u}_2 \right) - \left( \frac{\overrightarrow{u}_1}{2} + \frac{\overrightarrow{u}_2}{2} \right) \cos \left( \overrightarrow{u}_1, \overrightarrow{u}_2 \right) + \left( -\frac{\overrightarrow{u}_1}{2} - \frac{\overrightarrow{u}_2}{2} \right) \cos \left( \overrightarrow{u}_1, \overrightarrow{u}_2 \right) \end{pmatrix}$$
\[ \mathbf{J}_{\text{tot}} = R_1^2 \mathbf{v}_1 + R_2^2 \mathbf{v}_2 + R_1 R_2 \begin{pmatrix}
\left( \mathbf{v}_1 + \mathbf{v}_2 \right) \cos(\mathbf{v}_1, \mathbf{v}_2) + \left( \mathbf{v}_1 + \frac{\mathbf{u}_2}{2} \right) \cos(\mathbf{v}_1, \mathbf{u}_2) - \left( \mathbf{v}_1 - \frac{\mathbf{u}_2}{2} \right) \cos(\mathbf{v}_1, \mathbf{u}_2) \\
+ \left( \mathbf{u}_1 + \mathbf{v}_2 \right) \cos(\mathbf{u}_1, \mathbf{v}_2) - \left( \frac{\mathbf{u}_1}{2} + \mathbf{v}_2 \right) \cos(\mathbf{u}_1, \mathbf{v}_2) + \left( \frac{\mathbf{u}_1}{2} + \frac{\mathbf{u}_2}{2} \right) \cos(\mathbf{u}_1, \mathbf{u}_2) \\
- \left( \frac{\mathbf{u}_1}{2} - \frac{\mathbf{u}_2}{2} \right) \cos(\mathbf{u}_1, \mathbf{u}_2) - \left( \frac{\mathbf{u}_1}{2} + \frac{\mathbf{u}_2}{2} \right) \cos(\mathbf{u}_1, \mathbf{u}_2) + \left( \frac{\mathbf{u}_1}{2} - \frac{\mathbf{u}_2}{2} \right) \cos(\mathbf{u}_1, \mathbf{u}_2) \end{pmatrix} \]

\[ \Rightarrow \text{with, e.g., } R_1 R_2 \cos(\mathbf{v}_1, \mathbf{u}_2) = R_1 R_2 \cos(\frac{\pi}{2} + \phi) = -R_1 R_2 \sin(\phi), \text{ etc.:} \]

\[ \mathbf{J}_{\text{tot}} = P_1 \mathbf{v}_1 + P_2 \mathbf{v}_2 + \sqrt{P_1 P_2} (\mathbf{v}_1 + \mathbf{v}_2) \cos \phi + \sqrt{P_1 P_2} (\mathbf{u}_1 - \mathbf{u}_2) \sin \phi \]

genuinely qu. m. term
\[ \overline{J_{\text{tot}}} = R_1^2 \overline{v_1} + R_2^2 \overline{v_2} + R_1 R_2 \left[ \left( \overline{v_1} + \overline{u_2} \right) \cos(\overline{v_1}, \overline{v_2}) + \left( \overline{v_1} + \frac{\overline{u_2}}{2} \right) \cos(\overline{v_1}, \overline{u_2}) - \left( \overline{v_1} - \frac{\overline{u_2}}{2} \right) \cos(\overline{v_1}, \overline{u_2}) \right] \]

\[ + \left( \frac{\overline{u_1}}{2} + \overline{v_2} \right) \cos(\overline{u_1}, \overline{v_2}) - \left( \frac{\overline{u_1}}{2} + \overline{v_2} \right) \cos(\overline{u_1}, \overline{v_2}) + \left( \frac{\overline{u_1}}{2} + \frac{\overline{u_2}}{2} \right) \cos(\overline{u_1}, \overline{u_2}) \]

\[ - \left( \frac{\overline{u_1}}{2} - \frac{\overline{u_2}}{2} \right) \cos(\overline{u_1}, \overline{u_2}) - \left( \frac{\overline{u_1}}{2} + \frac{\overline{u_2}}{2} \right) \cos(\overline{u_1}, \overline{u_2}) + \left( \frac{\overline{u_1}}{2} - \frac{\overline{u_2}}{2} \right) \cos(\overline{u_1}, \overline{u_2}) \]

\[ \Rightarrow \text{with, e.g., } R_1 R_2 \cos(\overline{v_1}, \overline{u_2}) = R_1 R_2 \cos(\frac{\pi}{2} + \varphi) = -R_1 R_2 \sin(\varphi), \text{ etc.:} \]

\[ \overline{J_{\text{tot}}} = P_1 \overline{v_1} + P_2 \overline{v_2} + \sqrt{P_1 P_2} (\overline{v_1} + \overline{v_2}) \cos \varphi + \sqrt{P_1 P_2} (\overline{u_1} - \overline{u_2}) \sin \varphi \]

\[ \Rightarrow \text{compact formula for } N \text{ slits (Fussy et al., in preparation):} \]

\[ \overline{J_{\text{tot}}} = \sum_{i,j=1}^{3N} \left[ R_i R_j w_i \cos(\overline{w_i}, \overline{w_j}) \right] \]

with \( w_1 = \overline{v_1}, \ w_2 = \frac{\overline{u_1}}{2}, \ w_3 = -\frac{\overline{u_1}}{2}, \ w_4 = \overline{v_2}, \ldots, w_{3N} = -\frac{\overline{u}_N}{2} \)
Translation into “quantum language”:

inserting “the vocabulary” (i.e., including Madelung transformation, etc.)

\[
\vec{J}_{\text{tot}} = R_{\text{tot}}^2 \vec{v}_{\text{tot}} = (\psi_1 + \psi_2)^* (\psi_1 + \psi_2) \frac{1}{2} \left[ \frac{1}{m} \left( -i\hbar \nabla (\psi_1 + \psi_2) \right) + \frac{1}{m} \left( i\hbar \nabla (\psi_1 + \psi_2)^* \right) \right]
\]

\[
= -\frac{i\hbar}{2m} \left[ \Psi^* \nabla \Psi - \Psi \nabla \Psi^* \right] = \frac{1}{m} \text{Re} \left[ \Psi^* (-i\hbar \nabla) \Psi \right], \text{ with } \Psi = \psi_1 + \psi_2
\]

... exact qu. m. current!
Translation into “quantum language”:

inserting “the vocabulary” (i.e., including Madelung transformation, etc.)

\[
\mathbf{J}_{\text{tot}} = \frac{2}{m} \mathbf{v}_{\text{tot}} = (\psi_1 + \psi_2)^* (\psi_1 + \psi_2) \frac{1}{2} \left[ \frac{1}{m} \left( -i\hbar \frac{\nabla (\psi_1 + \psi_2)}{(\psi_1 + \psi_2)} \right) + \frac{1}{m} \left( i\hbar \frac{\nabla (\psi_1 + \psi_2)^*}{(\psi_1 + \psi_2)^*} \right) \right]
\]

\[
= -\frac{i\hbar}{2m} \left[ \psi^* \nabla \psi - \psi \nabla \psi^* \right] = \frac{1}{m} \text{Re} \left[ \psi^* (-i\hbar \nabla) \psi \right], \text{ with } \psi = \psi_1 + \psi_2
\]

Moreover:

\[
\frac{\partial P_{\text{tot}}}{\partial t} = -\nabla \cdot \mathbf{J}_{\text{tot}} = \frac{i\hbar}{2m} \nabla \left[ \psi^* \nabla \psi - \psi \nabla \psi^* \right] = \frac{\partial}{\partial t} \left( \psi^* \psi \right) = \psi^* \dot{\psi} + \psi \dot{\psi}^*
\]

Thus,

\[
i\hbar \left( \psi^* \dot{\psi} + \psi \dot{\psi}^* \right) = -\frac{\hbar^2}{2m} \left[ \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right], \text{ which is the difference of}
\]

\[
i\hbar \dot{\psi} \psi^* = \left( -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi \right) \psi^* \text{ and } -i\hbar \dot{\psi}^* \psi = \left( -\frac{\hbar^2}{2m} \nabla^2 \psi^* + V \psi^* \right) \psi
\]

\[
\rightarrow \quad i\hbar \dot{\psi} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi \quad \text{and c.c.}
\]
Quantum interference, weak dispersion
Quantum interference, strong dispersion
...plus different initial average velocities

$v_{x,2} = -4v_{x,1}$
...plus different initial spreadings

\[ \sigma_1 = 3 \sigma_2 \]
...with different initial probability densities

\[ P_1 = 2P_2 \]
Outlook:

1) entanglement: exploiting the nonlocality of DWF

2) complex situations: simulations! e.g., Ballistic diffusion: eddies

Fig. 2. Time development of the quantum potential showing constant x sections for barrier. Energy of incident Gaussian packet half barrier height.
...comparable phenomenology:
Summary:

- Explanatory framework for nonrelativistic quantum mechanics with the aid of “21st century classical mechanics”

- “Natural” drift in free quantum motion, due to sub-quantum anomalous (“ballistic”) diffusion; emergent: Gaussian dispersion

- de Broglie-Bohm theory: quantum potential as “mediator” (Holland) between classical potential $V$ and zones $V=0 \rightarrow$ “quantum force”; here: “natural” drift plus pure kinematics (momentum conservation on sub-quantum level)

- “the path excitation field”: interference at the double slit

- Outlook: particularly interesting w.r.t. entanglement